ANALYTIC SOLUTION OF QUARTIC AND CUBIC POLYNOMIALS

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REAL AND IMAGINARY ROOTS OF CUBIC AND QUARTIC POLYNOMIALS

1.1 INTRODUCTION

The analytic solution presented in this paper, may be used to find the real and imaginary roots of cubic and quartic polynomials in the form of

$$x^3 + a_2x^2 + a_1x + a_0 = 0 \qquad \text{(cubic)} \qquad (1)$$
$$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \qquad \text{(quartic)} \qquad (2)$$

Where, a_3, a_2, a_1 and a_0 are the real coefficients of the cubic and quartic polynomials. The leading coefficients are taken as 1. If they are not 1, they should be made 1 by dividing the entire equation by that coefficient.

1.2 COMPUTER PROGRAMS

The analytic solution for the determination of the real and imaginary roots of quartic and cubic polynomials and the computer programs based thereon were developed by the author and published by Hewlett-Packard, 100 N.E. Circle Blvd., Corvallis, Oregon 97330, USA. The relevant HP Users' Libraries and the published material are detailed as follows

a) Real roots only

HP-67/97/41 Users' Library. Solution was registered under category No. L450, Catalogue No. 02785C4, April 29, 1984. The program was developed for HP 41C calculators.

HP-75 Users' Library. Solution was registered under category No. L450, Catalogue No. 7500154, May 17, 1984. The program was developed for HP-75C computers.

b) Real and imaginary roots

HP-75 Users' Library. Solution was registered under category No. L450, catalogue # 75 00154, March 3, 1986. The program was developed for HP-75C computers.

1.3 REAL AND IMAGINARY ROOTS OF CUBIC POLYNOMIALS

1.3.1 Rreal roots

A general form of cubic polynomials may be written as

$$y^3 + b_2y^2 + b_1y + b_0 = 0 \tag{3}$$

where, b_2, b_1 and b_0 are real coefficients

Let $y = x - \frac{b_2}{3}$, where x is avariable.Then,

$$x^3 + \left(b_1 - \frac{b_2^{\,2}}{3} \right) x + b_0 - \frac{b_1 b_2}{3} + \frac{2b_2^{\,3}}{27} = 0 \tag{4}$$

let $a_1 = b_1 - \frac{b_2^{\,2}}{3}$ and $a_0 = b_0 - \frac{b_1 b_2}{3} + \frac{2b_2^{\,3}}{27}$ then equation (4) becomes

$$x^3 + a_1x + a_0 = 0 \tag{5}$$

a) Trigonometric solution

Constructed in Fig. 1, are a circle and an angle at its centre equal 3ß. Angle 3ß is divided into three equal angles, each equal ß. Cords opposite to angles ß and to 3ß are constructed forming similar triangles. From the formed similar triangles, the following relations may be written

$$\frac{BC}{OB} = \frac{CD}{BC} \quad \text{and} \quad \frac{BC}{OD} = \frac{CD}{FD} \quad \text{hence } x^2 = rz \text{ and } \quad \frac{x}{r - z} = \frac{z}{c - 2x}$$

where, $c = AB$. By substitution, we get

$$x^3 - 3r^2 + r^2c = 0 \tag{6}$$

Equation (5) = equation (6) if and only if $a_1 = - 3r^2$ and $a_0 = r^2c$

Therefore,

$$r = \sqrt{-\frac{a_1}{3}} \quad \text{and} \quad c = -\frac{3a_0}{a_1} \tag{7}$$

From trigonometry, the following relations may be derived

$$x = 2r\sin\frac{\beta}{2} \quad \text{and} \quad c = 2r\sin\frac{3}{2}\beta \tag{8}$$

hence,

$$-\frac{3a_0}{a_1} = 2\sqrt{\frac{-a_1}{3}}\sin\frac{3}{2}\beta$$

Therefore,

$$\beta = \frac{2}{3}\sin^{-1}\frac{-3\sqrt{3}a_0}{2a_1\sqrt{-a_1}} \tag{9}$$

The first real root x_1 of equation (5) is

$$x_1 = 2\sqrt{\frac{-a_1}{3}}\sin\left(\frac{1}{3}\sin^{-1}\frac{-3\sqrt{3}a_0}{2a_1\sqrt{-a_1}}\right) \tag{10}$$

The other two real roots x_2 and x_3 may then be determined as follows

$$x_2 \,\&\, x_3 = \frac{-x_1 \pm \sqrt{x_1^{\,2} - 4\left(x_1^{\,2} + a_1\right)}}{2} \tag{11}$$

This solution may be applied if and only if

$$a_1 < 0 \quad \text{and} \quad \left|-3\sqrt{3}a_0\right| \le \left|2a_1\sqrt{-a_1}\right| \tag{12}$$

Whence, the roots of equation (3), y_1, y_2 and y_3 may be determined as follows

$$y_1 = x_1 - \frac{b_2}{3}, \qquad y_2 = x_2 - \frac{b_2}{3} \text{ and } \qquad y_3 = x_3 - \frac{b_2}{3} \tag{13}$$

b) Algebraic solution

If the conditions in equation (12) are not satisfied then the cubic polynomial has one real root. The method for finding it is essentially that given by Hudde in 1650.

By transformation, we get

$x = z - \frac{a_1}{3z}$, where z is a variable.

Then, by substitution, equation (5) becomes

$$z^6 + a_0 z^3 - \frac{a_1^{\,3}}{27} = 0 \tag{14}$$

hence,

$$z^3 = -\frac{a_0}{2} \pm \sqrt{\frac{a_0^{\,2}}{4} + \frac{a_1^{\,3}}{27}} \qquad \text{and} \qquad z = \left\{-\frac{a_0}{2} \pm \sqrt{\frac{a_0^{\,2}}{4} + \frac{a_1^{\,3}}{27}}\right\}^{\frac{1}{3}}$$

By substitution for x, we get

$$x = z - \frac{a_1}{3z} = \left\{-\frac{a_0}{2} \pm \sqrt{\frac{a_0^{\,2}}{4} + \frac{a_1^{\,3}}{27}}\right\}^{\frac{1}{3}} - \frac{a_1}{3}\left\{-\frac{a_0}{2} \pm \sqrt{\frac{a_0^{\,2}}{4} + \frac{a_1^{\,3}}{27}}\right\}^{\frac{-1}{3}} \tag{15}$$

whence, the real root y_1 of equation (3), may now be determined as

$$y_1 = x - \frac{b_2}{3} \qquad (16)$$

c) Geometric solution

Appendix A offers a solution in Euclid's geometry, for finding the real roots of cubic polynomials that satisfy the conditions of equation (12).

1.3.2 Imaginary roots

If a real root is found by equation (15), the cubic polynomial has then two more imaginary roots that may be determined as follows

Divide equation (3) by $y - y_1$

$$\frac{y^3 + b_2 y^2 + b_1 y + b_0}{y - y_1} = y^2 + (b_2 + y_1)y + b_1 + b_2 y_1 + y_1^{\,2} + \frac{b_0 + b_1 y_1 + b_2 y_1^{\,2} + y_1^{\,3}}{y - y_1}$$

Since y_1 is a real root then

$$b_0 + b_1 y_1 + b_2 y_1^{\,2} + y_1^{\,3} = 0$$

and,

$$y^2 + (b_2 + y_1)y + b_1 + b_2 y_1 + y_1^{\,2} = 0$$

The imaginary roots y_{i2} and y_{i3} are then determined as follows

$$y_{i2} \,\&\, y_{i3} = \frac{-(b_2 + y_1)}{2} \pm \frac{\sqrt{\left|(b_2 + y_1)^2 - 4\left(b_1 + b_2 y_1 + y_1^{\,2}\right)\right|}}{2}\, j \qquad (17)$$

where $j = \sqrt{-1}$.

Note that a cubic polynomial might have the following number of roots

Three real roots, or
One real and two imaginary roots

1.4 REAL AND IMAGINARY ROOTS OF QUARTIC POLYNOMIALS

1.4.1 Real roots

A general form of quartic polynomial is written as follows [1]

$$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \qquad (18)$$

Where, a_3, a_2, a_1 and a_0 are the real coefficients. The leading coefficient is taken as 1. If it is not 1, it should be made 1 by dividing the entire equation by that coefficient.

Equation (18) may be factored into two quadratic polynomials as follows

$$\left\{x^2 + (A + C)x + B + D\right\}\left\{x^2 + (A - C)x + B - D\right\} = 0 \qquad (19)$$

Where, A, B, C and D are so far unknown coefficients for the quadratic expressions.

Multiplying the quadratic expressions by each other leads to

$$x^4 + 2Ax^3 + \left(A^2 - C^2 + 2B\right)x^2 + (2AB - 2CD)x + B^2 - D^2 = 0 \qquad (20)$$

Equation (20) = equation (18) if and only if

$$a_3 = 2A \qquad (21)$$
$$a_2 = A^2 + 2B - C^2 \qquad (22)$$
$$a_1 = 2(AB - CD) \qquad (23)$$
$$a_0 = B^2 - D^2 \qquad (24)$$

Solving simultaneously equations (21) to (24) leads to

$$A = \frac{a_3}{2} \qquad (25)$$

$$B^3 - \frac{a_2}{2}B^2 + \left(\frac{a_1a_3}{4} - a_0\right)B + \frac{a_0a_2}{2} - \frac{a_1^{\,2}}{8} - \frac{a_0a_3^{\,2}}{8} = 0 \qquad (26)$$

$$C = \pm\sqrt{\frac{a_3^{\,2}}{4} - a_2 + 2B} \qquad (27)$$

$$D = \pm\sqrt{B^2 - a_0} \qquad (28)$$

Equation (26) is a cubic equation in its general form. To solve it let $B = y + \frac{a_2}{6}$, where y is a variable. This leads to the following equation

$$y^3+\left(\frac{a_1a_3}{4}-\frac{a_2^{\,2}}{12}-a_0\right)y+\frac{a_1a_2a_3}{24}+\frac{a_0a_2}{3}-\frac{a_1^{\,2}}{8}-\frac{a_0a_3^{\,2}}{8}-\frac{a_2^{\,3}}{108}=0 \qquad (29)$$

Let $\quad q=\frac{a_1a_3}{4}-\frac{a_2^{\,2}}{12}-a_0 \quad$ and $\quad r=\frac{a_1a_2a_3}{24}+\frac{a_0a_2}{3}-\frac{a_1^{\,2}}{8}-\frac{a_0a_3^{\,2}}{8}-\frac{a_2^{\,3}}{108}$

then, an algebraic solution for B gives

$$B=y+\frac{a_2}{6}=\left\{-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right\}^{\frac{1}{3}}-\frac{q}{3}\left\{-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}\right\}^{\frac{-1}{3}}+\frac{a_2}{6} \qquad (30)$$

If the quantity under the square root is negative then a trigonometric solution for B gives

$$B=\frac{a_2}{6}-2\sqrt{\frac{|q|}{3}}\cos\frac{\beta}{2} \qquad (31)$$

where,

$$\beta=\cos^{-1}\frac{\sqrt{27}\,r}{2|q|^{\frac{3}{2}}}$$

The determination of B leads to the determination of C and D in equations (27) and (28). Thus the previously unknown coefficients A, B, C and D become now known and are expressed in terms of a_3, a_2, a_1 and a_0. The real roots, may therefore be determined as follows

The 1st and 2nd roots x_1 and x_2, are

$$x_1\,\&\,x_2=\frac{-(A+C)\pm\sqrt{(A+C)^2-4(B+D)}}{2} \qquad (32)$$

The 3rd and 4th roots x_3 and x_4, are

$$x_3\,\&\,x_4=\frac{-(A-C)\pm\sqrt{(A-C)^2-4(B-D)}}{2} \qquad (33)$$

1.4.2 Imaginary roots

If the quantities under the square roots in equations (27) and (28) are negative, then the coefficients C and D become imaginary. No solution is offered for quartic polynomials with imaginary coefficients. Coefficients A, B, C and D must all be real.

If the quantities under the square roots in equations (32) and (33) are negative, then the quartic polynomial has imaginary roots that may be determined as follows

The 1st and 2nd imaginary roots x_{i1} and x_{i2}, are

$$x_{i1} \& x_{i2} = \frac{-(A+C)}{2} \pm \frac{\sqrt{\left|(A+C)^2 - 4(B+D)\right|}}{2} j \tag{34}$$

The 3rd and 4th imaginary roots x_{i3} and x_{i4}, are

$$x_{i3} \& x_{i4} = \frac{-(A-C)}{2} \pm \frac{\sqrt{\left|(A-C)^2 - 4(B-D)\right|}}{2} j \tag{35}$$

where $j = \sqrt{-1}$.

Note that a qurtic polynomial might have the following number of roots

Four real roots, or
Two real and two imaginary roots, or
Four imaginary roots

REFERENCES

1 HP - 41C Math Pac, Hewlett-Packard, 100 N.E. Circle Blvd., Corvallis, Oregon 97330, USA, 1979.

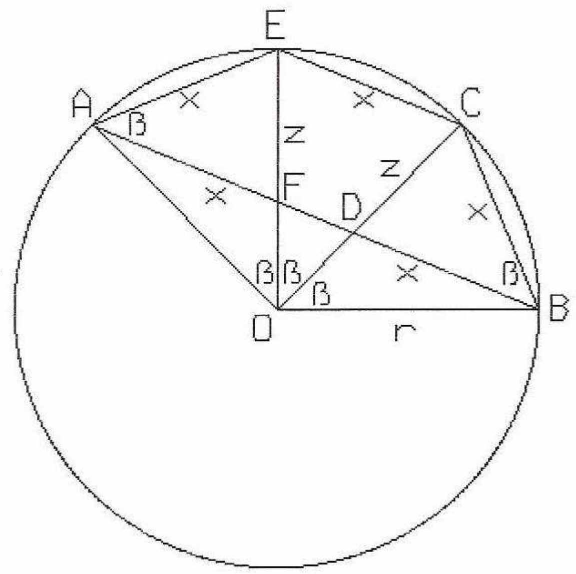

Fig 1 Geometry of similar triangles